Odomutants and flexibility results for quantitative orbit equivalence

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Optimality of Belinskaya's theorem for odometers:

Theorem (C. 2025+)

Let $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear map and let S be an odometer. Then there exists $T \in \mathrm{Aut}(X, \mu)$ such that T and S are φ-OE but not flip-conjugate.

Optimality of Kerr and Li's theorem:

Theorem (C. 2025+)

Let S be the universal odometer, let $\alpha > 0$ or $\alpha = +\infty$. There exists $T \in \mathrm{Aut}(X, \mu)$ such that:

- $\mathrm{h}_\mu(T) = \alpha$;
- *T and S are $\log^\beta$-OE for all $\beta < 1$.*

Common construction:

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Common construction:

Odometer S

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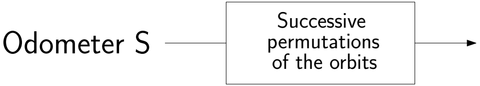
Common construction:



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Common construction:

Odometer S —— $\boxed{\text{Successive permutations of the orbits}}$ ——→ ☢ Odomutant T ☢

Odometer S $\rightsquigarrow$ ☢ Odomutant T ☢

- Let's define odomutants;
- prove a weaker version of the theorem on entropy:

Theorem (C. 2025+, weaker version)

Let S be the universal odometer. There exists $T \in \mathrm{Aut}(X, \mu)$ such that:

- $\mathrm{h}_\mu(T) > 0$;
- *T and S are $\log^\beta$-OE for all $\beta < 1$.*

- and sketch the proof of the other theorem (optimality of Belinskaya's theorem for odometers)

Another definition of odometer.

Combinatorial structure:

Definition (n-cylinders)

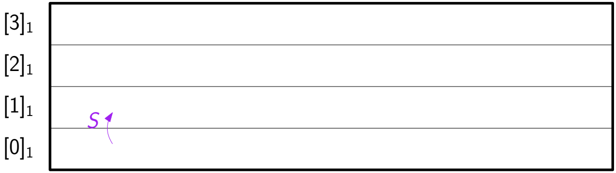
Given $i_0 \in \{0, 1, \dots, q_0 - 1\}, \dots, i_{n-1} \in \{0, 1, \dots, q_{n-1} - 1\}$,

$$[i_0, \dots, i_{n-1}]_n \coloneqq \{(x_i)_{n \geq 0} \in X \mid x_0 = i_0, \dots, x_{n-1} = i_{n-1}\}.$$

$q_0 = 4$

$[3]_1$	
$[2]_1$	
$[1]_1$	
$[0]_1$	

$q_0 = 4$



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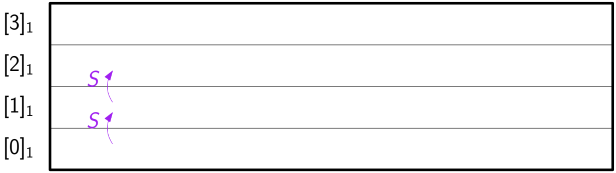

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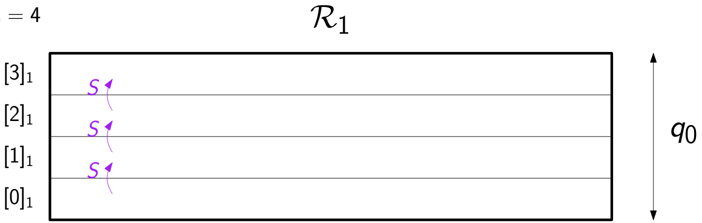

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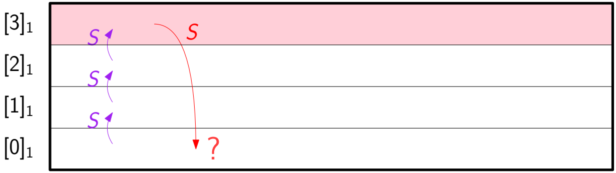

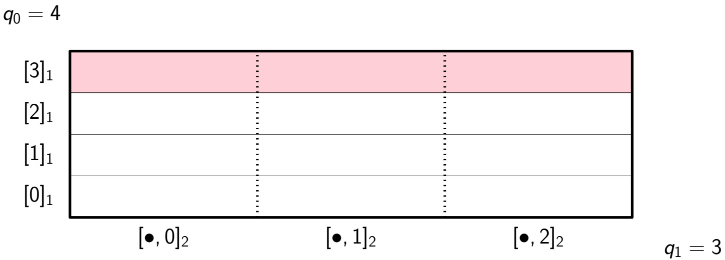
$q_0 = 4$
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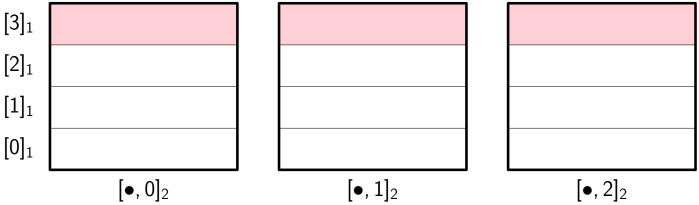
$[\bullet, 1]_2$
$[\bullet, 2]_2$
$q_1 = 3$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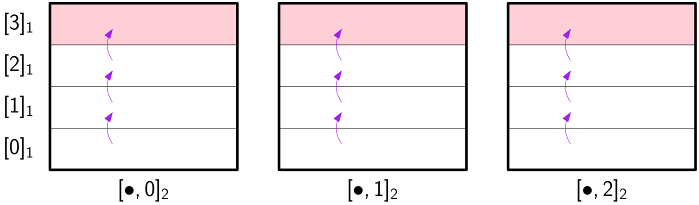
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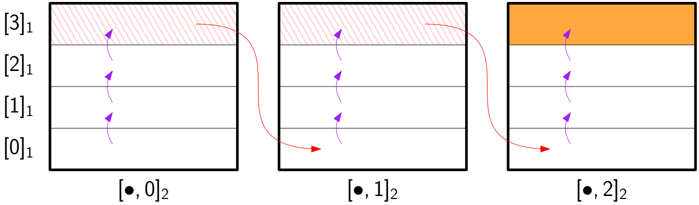

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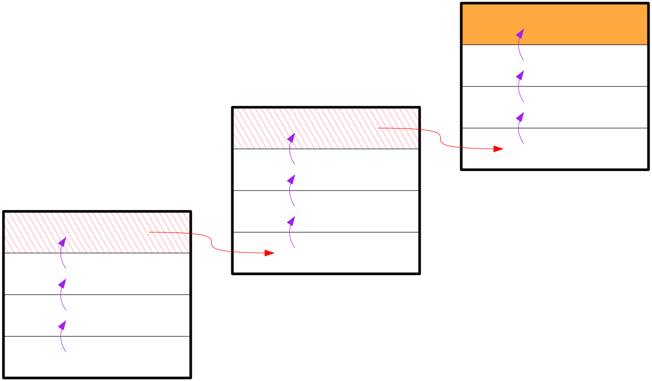

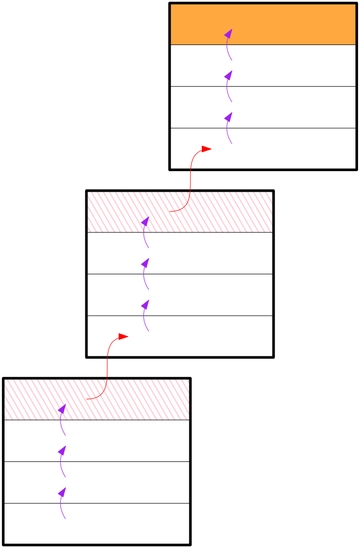

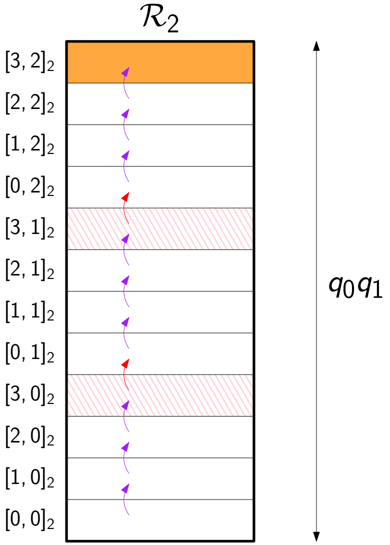
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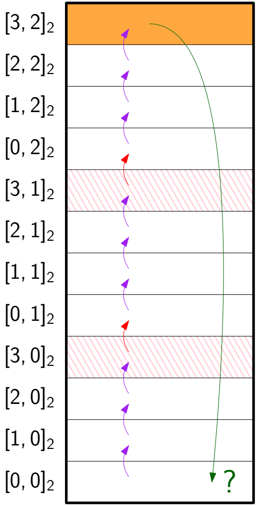
$[0,0]_2$
$q_0 q_1$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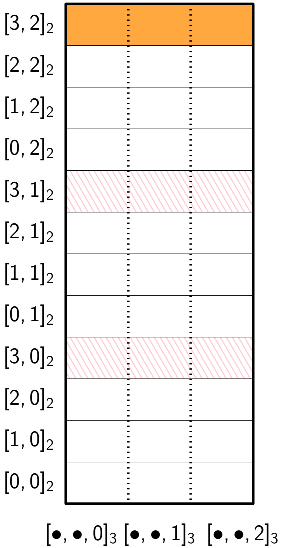
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$[2,0]_2$
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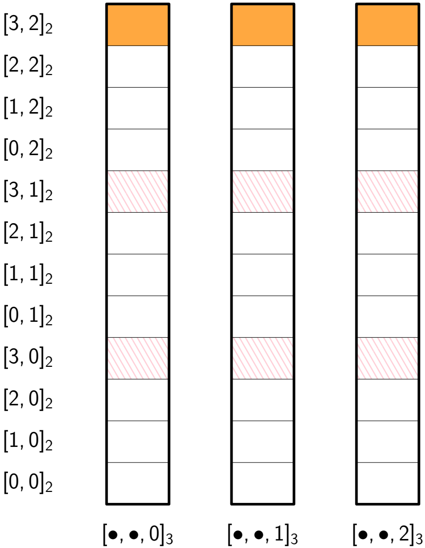
$[0,0]_2$
$[\bullet,\bullet,0]_3$ $[\bullet,\bullet,1]_3$ $[\bullet,\bullet,2]_3$

$q_2 = 3$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$[1, 2]_2$
$[0, 2]_2$
$[3, 1]_2$
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$[1, 1]_2$
$[0, 1]_2$
$[3, 0]_2$
$[2, 0]_2$
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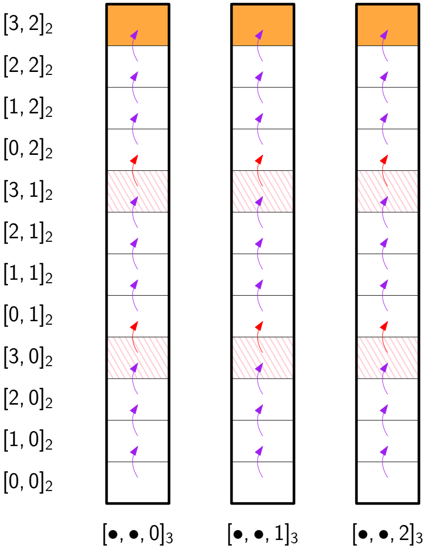
$[\bullet, \bullet, 1]_3$
$[\bullet, \bullet, 2]_3$


$[3,2]_2$
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$[3,1]_2$
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$[0,1]_2$
$[3,0]_2$
$[2,0]_2$
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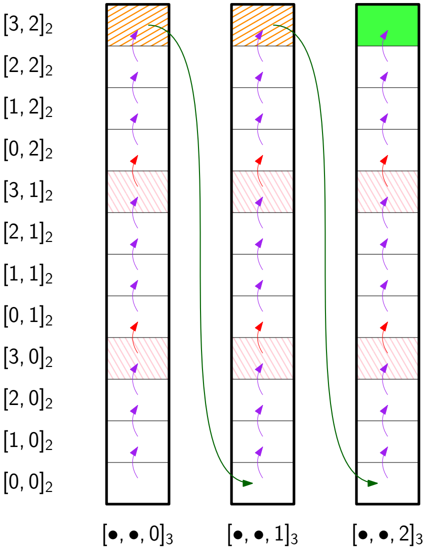
$[\bullet,\bullet,2]_3$



$[3,2]_2$
$[2,2]_2$
$[1,2]_2$
$[0,2]_2$
$[3,1]_2$
$[2,1]_2$
$[1,1]_2$
$[0,1]_2$
$[3,0]_2$
$[2,0]_2$
$[1,0]_2$
$[0,0]_2$
$[\bullet,\bullet,0]_3$
$[\bullet,\bullet,1]_3$
$[\bullet,\bullet,2]_3$

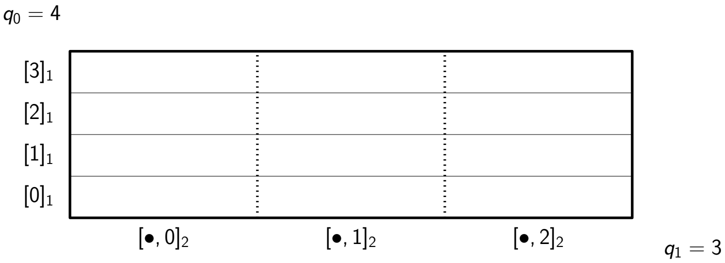
Odometer:

$$\begin{tikzcd}
 & {[1]_1} \arrow[r, bend left, "S"] & {[2]_1} \arrow[dr, bend left, no head] & \\
{[0]_1} \arrow[ur, bend left, "S"] & & & \vdots \arrow[dl, bend left, "S"] \\
 & {[q_0 - 1]_1} \arrow[ul, bend left, "S"] & {[q_0 - 2]_1} \arrow[l, bend left, "S"] &
\end{tikzcd}$$

With an odomutant: less “predictable” dynamics

$q_0 = 4$

$[3]_1$	
$[2]_1$	
$[1]_1$	
$[0]_1$	



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$[3]_1$
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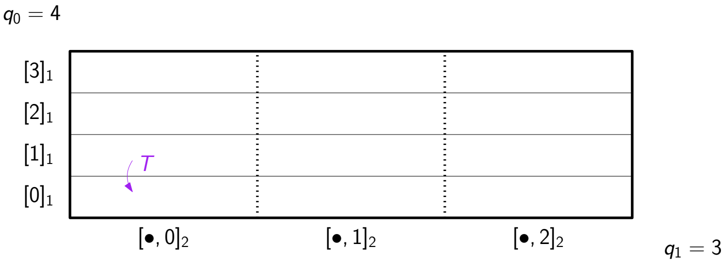
$q_0 = 4$
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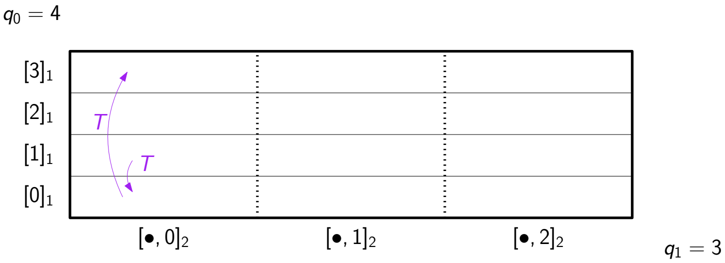
$[\bullet, 2]_2$
$q_1 = 3$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$[\bullet, 1]_2$
$[\bullet, 2]_2$
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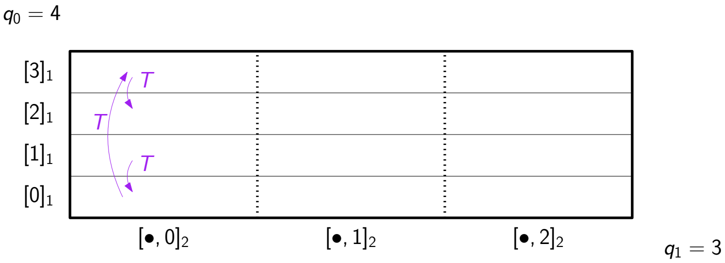
$q_0 = 4$
$[3]_1$
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$[\bullet, 0]_2$
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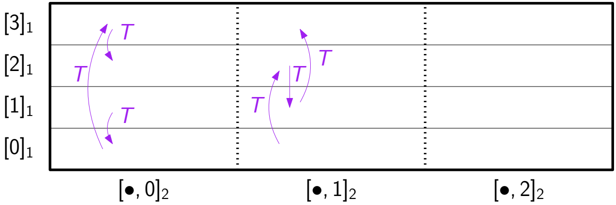
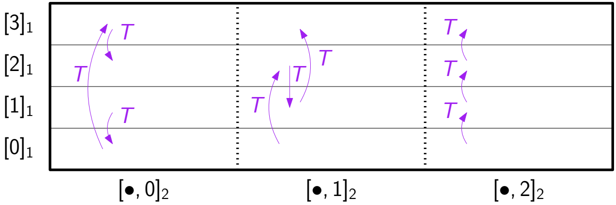
$q_0 = 4$



$q_1 = 3$

$q_0 = 4$



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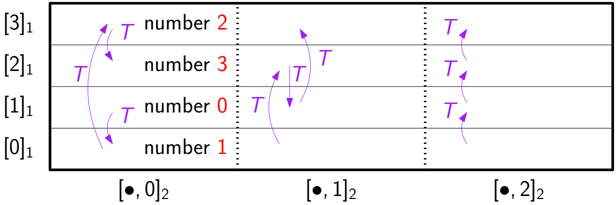
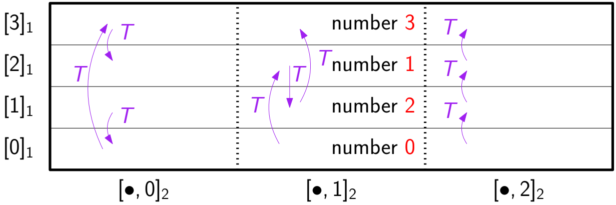


$q_1 = 3$

Permutations :
$0 \mapsto 1$
$1 \mapsto 0$
$2 \mapsto 3$
$3 \mapsto 2$

$q_0 = 4$



$q_1 = 3$

Permutations :

$[\bullet, 0]_2$	$[\bullet, 1]_2$
$0 \mapsto 1$	$0 \mapsto 0$
$1 \mapsto 0$	$1 \mapsto 2$
$2 \mapsto 3$	$2 \mapsto 1$
$3 \mapsto 2$	$3 \mapsto 3$

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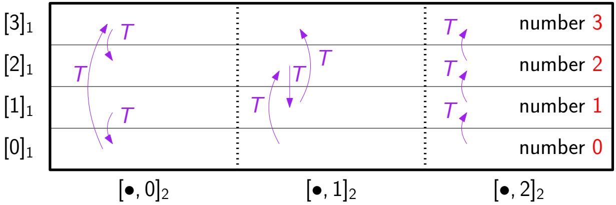

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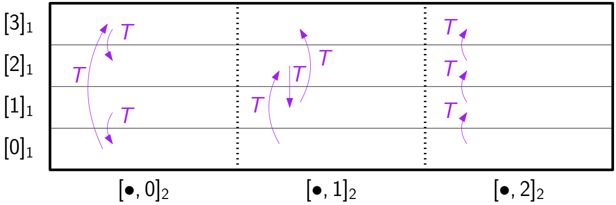
$0 \mapsto 1$	$0 \mapsto 0$	$0 \mapsto 0$
$1 \mapsto 0$	$1 \mapsto 2$	$1 \mapsto 1$
$2 \mapsto 3$	$2 \mapsto 1$	$2 \mapsto 2$
$3 \mapsto 2$	$3 \mapsto 3$	$3 \mapsto 3$

$$\sigma_{x_1}^{(0)} \longrightarrow \text{step } 0$$

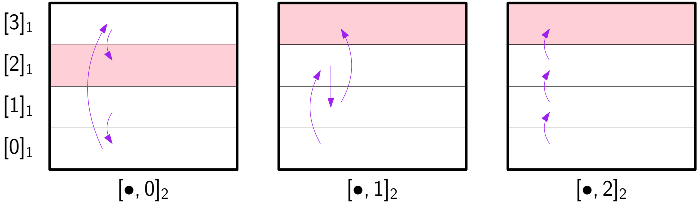
$$\sigma_{x_1}^{(0)} \longrightarrow \text{subcolumn } [\bullet, x_1]_2$$

$q_0 = 4$



$q_1 = 3$

Permutations : $\sigma_0^{(0)}$: $0 \mapsto 1$, $1 \mapsto 0$, $2 \mapsto 3$, $3 \mapsto 2$ $\quad$ $\sigma_1^{(0)}$: $0 \mapsto 0$, $1 \mapsto 2$, $2 \mapsto 1$, $3 \mapsto 3$ $\quad$ $\sigma_2^{(0)}$: $0 \mapsto 0$, $1 \mapsto 1$, $2 \mapsto 2$, $3 \mapsto 3$


$[3]_1$
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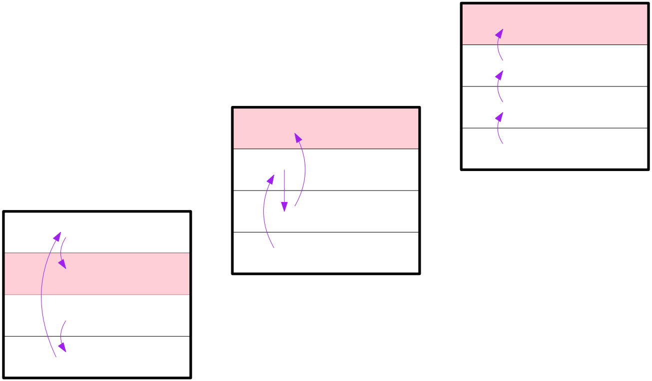

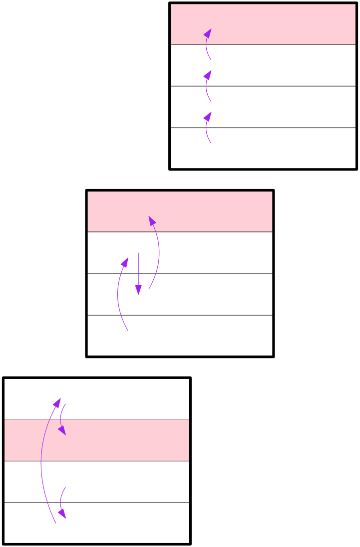

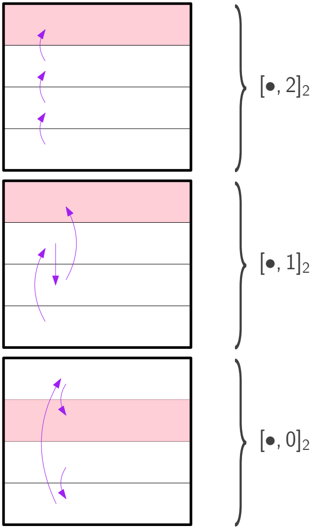
$[\bullet, 2]_2$
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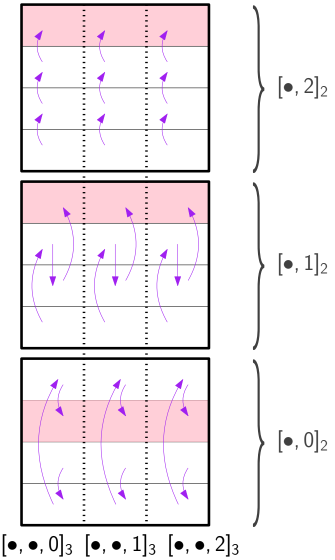
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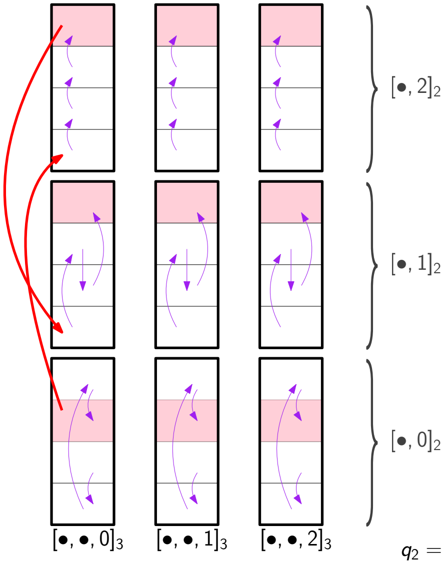
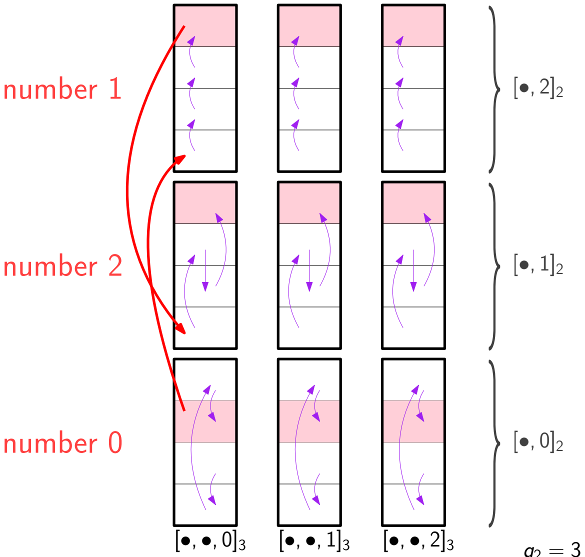


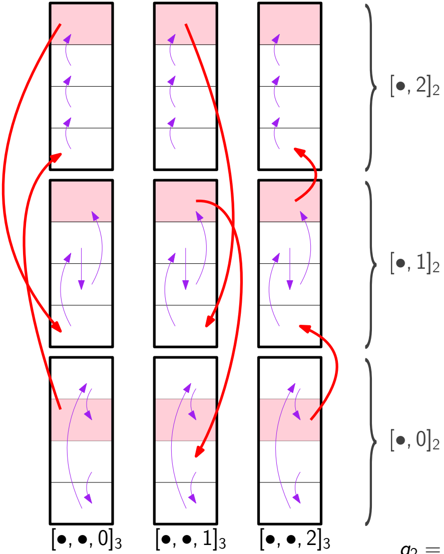
$[\bullet, 2]_2$
$[\bullet, 1]_2$
$[\bullet, 0]_2$
$[\bullet, \bullet, 0]_3$
$[\bullet, \bullet, 1]_3$
$[\bullet, \bullet, 2]_3$

$q_2 = 3$



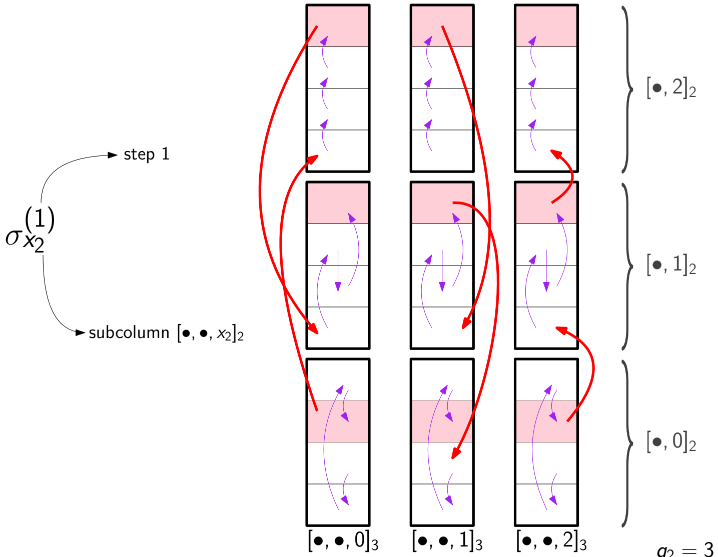
Permutations :

$0 \mapsto 0$
$1 \mapsto 2$
$2 \mapsto 1$



$q_2 = 3$

Permutations :	$0 \mapsto 0$	$0 \mapsto 2$	$0 \mapsto 0$
	$1 \mapsto 2$	$1 \mapsto 1$	$1 \mapsto 1$
	$2 \mapsto 1$	$2 \mapsto 0$	$2 \mapsto 2$



Permutations :

$\sigma_0^{(1)}: \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 2 \\ 2 \mapsto 1 \end{array}$ $\sigma_1^{(1)}: \begin{array}{l} 0 \mapsto 2 \\ 1 \mapsto 1 \\ 2 \mapsto 0 \end{array}$ $\sigma_2^{(1)}: \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 1 \\ 2 \mapsto 2 \end{array}$

An odomutant T is the data of

- **integers $\boldsymbol{q_0, q_1, q_2} \ldots \geq 2$**
  and the odometer S on $X = \prod_{n \geq 0} \{0, 1, \ldots, q_n - 1\}$;

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- for every $n \geq 0$:
  q_{n+1} **permutations** of $\{0, 1, \ldots, q_n - 1\}$: $\boldsymbol{\sigma_0^{(n)}, \sigma_1^{(n)}, \ldots, \sigma_{q_{n+1}-1}^{(n)}}$.

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At step n:

x_{n+1} $(n+1)$-th coordinate $\in \{0, \ldots, q_{n+1} - 1\}$
represents the block $[\bullet, \ldots, \bullet, \bullet, x_{n+1}]_{n+2}$

$\downarrow$

$\sigma_{x_{n+1}}^{(n)}$ permutation on the n-th coordinate $\in \{0, \ldots, q_n - 1\}$

An odomutant T is the data of

- **integers** $\boldsymbol{q_0, q_1, q_2} \ldots \geq 2$
  and the odometer S on $X = \prod_{n \geq 0} \{0, 1, \ldots, q_n - 1\}$;
- for every $n \geq 0$:
  q_{n+1} **permutations** of $\{0, 1, \ldots, q_n - 1\}$: $\boldsymbol{\sigma_0^{(n)}, \sigma_1^{(n)}, \ldots, \sigma_{q_{n+1}-1}^{(n)}}$.

At step n:

x_{n+1} $(n+1)$-th coordinate $\in \{0, \ldots, q_{n+1} - 1\}$
represents the block $[\bullet, \ldots, \bullet, \bullet, x_{n+1}]_{n+2}$

$\downarrow$

$\sigma_{x_{n+1}}^{(n)}$ permutation on the n-th coordinate $\in \{0, \ldots, q_n - 1\}$
$\rightarrow$ permutation of sub-blocks

$$[\bullet, \ldots, \bullet, 0, x_{n+1}]_{n+2},$$
$$[\bullet, \ldots, \bullet, 1, x_{n+1}]_{n+2},$$
$$\vdots$$
$$[\bullet, \ldots, \bullet, q_n, x_{n+1}]_{n+2}$$

$$
\begin{array}{rccl}
\psi_n \colon & X & \longrightarrow & X \\
& (x_n)_{n\geq 0} & \longmapsto & \left(\sigma_{x_1}^{(0)}(x_0),\, \sigma_{x_2}^{(1)}(x_1), \dots,\, \sigma_{x_{n+1}}^{(n)}(x_n), x_{n+1}, x_{n+2}, \dots\right)
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Definition (Odomutant T associated to the odometer S)

$$T\colon \{x \in X \mid \psi(x) \neq (q_0 - 1, q_1 - 1, \ldots)\} \longrightarrow \{x \in X \mid \psi(x) \neq (0, 0, \ldots)\}$$

$$\boldsymbol{Tx = \lim_{n\to+\infty} \psi_n^{-1} S \psi_n(x)}$$

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- $T \in \mathrm{Aut}(X, \mu)$;
- T and S have the same measure (up to a null set);
- $\psi \circ T = S \circ \psi$ almost everywhere,
  so T factorizes onto S.

Proof of:

Theorem (C. 2025+, weaker version)

Let S be the universal odometer. There exists $T \in \mathrm{Aut}(X, \mu)$ such that:

- $\mathrm{h}_\mu(T) > 0$;
- *T and S are* $\log^\beta$*-OE for all $\beta < 1$.*

Calculation of $\mathrm{h}_\mu(T)$:

Proof of:

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Calculation of $\mathrm{h}_\mu(T)$:

- easier to compute topological entropy
  $\rightarrow$ $T\colon X \rightarrow X$ has to be continuous X
- apply the variational principle
  $\rightarrow$ If T is uniquely ergodic, then $\mathrm{h}_{\mathrm{top}}(T) = \mathrm{h}_\mu(T)$

HYPOTHESIS 1:

For every $n \geq 0$, 0 et $q_n - 1$ are fixed point of the permutations $\sigma_0^{(n)}, \sigma_1^{(n)}, \ldots, \sigma_{q_{n+1}-1}^{(n)}$.

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Property

- *$T\colon X \to X$ is a homeomorphism (so is continuous).*
- *For every $x \in X$ (not only "up to a null set"),*
  $\mathrm{Orb}_T(x) = \mathrm{Orb}_S(x)$ *(so T is uniquely ergodic).*

Goal:

- find the parameters:
  - the integers $\boldsymbol{q_n}$ (and the underlying odometer S);
  - and the permutations $\boldsymbol{\sigma_i^{(n)}}$ (satisfying HYPOTHESIS 1),

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  such that $\mathrm{h}_{\mathrm{top}}(T) > 0$;
- ...in the hope of getting an orbit equivalence which is $\log^{\beta}$-integrable for every $\beta < 1$.

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Given an open cover $\mathcal{U}$ of X,

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$$\mathrm{h_{top}}(T) \geq \lim_{n \to +\infty} \frac{\log(\text{number of } n\text{-words } (i_0,\ldots,i_{n-1}) \text{ obtained from 1-cylinders})}{n}$$

Reminder on HYPOTHESIS 1:
For every $n \geq 0$, 0 and $q_n - 1$ are fixed points of the permutations $\sigma_0^{(n)}, \ldots, \sigma_{q_{n+1}-1}^{(n)}$.

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For every $n \geq 0$, $\sigma_0^{(n)}, \ldots, \sigma_{q_{n+1}-1}^{(n)}$ describe all the permutations of $\{0, 1, \ldots, q_n - 1\}$ fixing 0 and $q_n - 1$, and are pairwise distinct.

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Proposition

With HYPOTHESIS 2, we produce a lot of words !

$$\mathrm{h}_{\mathrm{top}}(T) \geq (\log q_0) - C \text{ where } C \text{ is a constant.}$$

If q_0 is large enough, then $\mathrm{h}_{\mathrm{top}}(T) > 0$.

Proof of the proposition:

1. We prove that

$$\text{"number of } h_n\text{-words obtained from 1-cylinders"} \geq q_n$$

   where $h_{n+1} = q_0 q_1 \ldots q_n$ is the height of the towers at step n.

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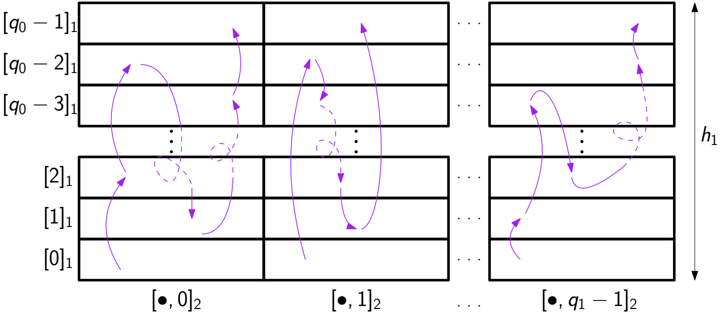
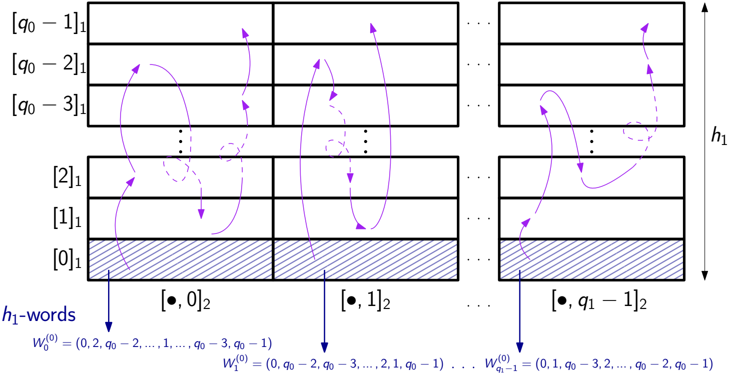
2. $\mathrm{h}_{\mathrm{top}}(T) \geq \lim_{n \to +\infty} \frac{\log(\text{number of } n\text{-words obtained from 1-cylinders})}{n}$,
   so $\mathrm{h}_{\mathrm{top}}(T) \geq \lim_{n \to +\infty} \frac{\log q_n}{h_n}$.

   To get $\frac{\log q_n}{h_n} \geq (\log q_0) - C$, we successively use
   - $q_{i+1} = (q_i - 2)!$;
   - $\log(q_i!) \geq q_i \log q_i$;
   - $h_{i+1} = q_i h_i$.


$[q_0-1]_1$
$[q_0-2]_1$
$[q_0-3]_1$
$[2]_1$
$[1]_1$
$[0]_1$
$[\bullet,0]_2$
$[\bullet,1]_2$
$[\bullet,q_1-1]_2$
h_1

...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h_1
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h_1-words
$W_0^{(0)} = (0, 2, q_0-2, \dots, 1, \dots, q_0-3, q_0-1)$
$W_1^{(0)} = (0, q_0-2, q_0-3, \dots, 2, 1, q_0-1)$
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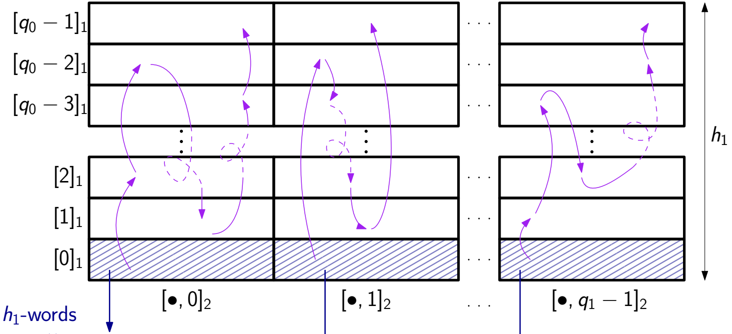

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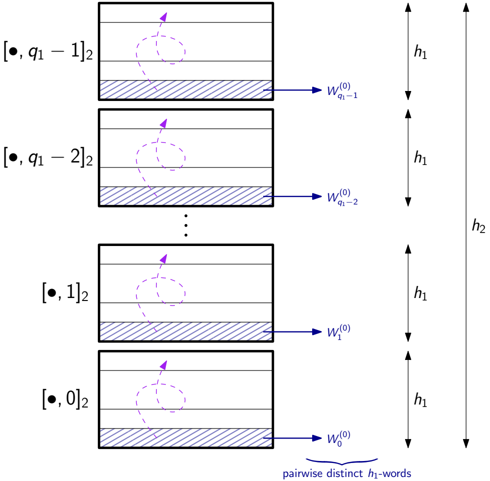
Pairwise distinct permutations $\Rightarrow$ at least q_1 h_1-words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h_1
h_1
h_1
h_1
h_2
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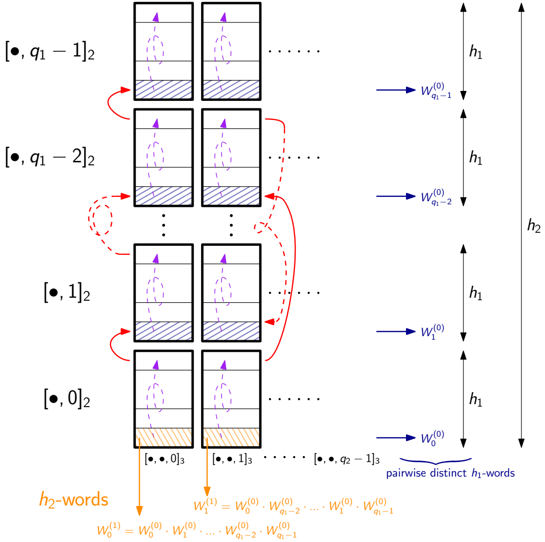

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pairwise distinct h_1-words
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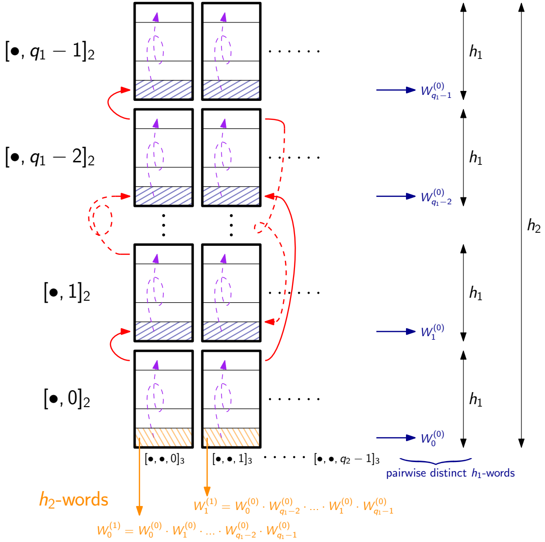

$$\forall i \in \{0, \ldots, q_2-1\},\ W_i^{(1)} = W_0^{(0)} \cdot W^{(0)}_{\sigma_i^{(1)}(1)} \cdot W^{(0)}_{\sigma_i^{(1)}(2)} \cdot \ldots \cdot W^{(0)}_{\sigma_i^{(1)}(q_1-2)} \cdot W^{(0)}_{q_1-1}$$

Pairwise distinct permutations and pairwise distincts subwords $W_j^{(0)}$

$\Rightarrow$ at least q_2 h_2-words

Proposition

If $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is non-decreasing, then* $\int_X \varphi(|c_T(x)|)\mathrm{d}\mu(x)$ *and* $\int_X \varphi(|c_S(x)|)\mathrm{d}\mu(x)$ *are bounded above by* $\sum_{n\geq 0} \frac{\varphi(q_0 \dots q_{n+1})}{q_0 \dots q_n}$.

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Corollary

With $q_{n+1} = (q_n - 2)!$, *the cocycles are* $\log^\beta$*-integrable for every* $\beta < 1$.

This concludes the proof of the theorem on entropy (the weaker version).

Topological version:

Theorem (C. 2025+)

Let S be the universal odometer, let $\alpha > 0$ or $\alpha = +\infty$. There exists a Cantor minimal homeomorphism T such that:

- $\mathrm{h}_{\mathrm{top}}(T) = \alpha$;
- *T and S are **strongly orbit equivalent**, with* $\log^{\beta}$*-integrable cocycles for every $\beta < 1$.*

Strong orbit equivalence: the cocycles each have at most one point of discontinuity.

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Generalization of:

Theorem (Boyle–Handelman 1994)

Let S the dyadic odometer, let $\alpha > 0$ or $\alpha = +\infty$. There exists a Cantor minimal homeomorphim T such that:

- $\mathrm{h}_{\mathrm{top}}(T) = \alpha$;
- *T and S are **strongly orbit equivalent**.*

Sketch of proof of:

Theorem (C. 2025+)

Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear map and let S be an odometer. Then there exists $T \in \mathrm{Aut}(X,\mu)$ such that T and S are φ-OE but not flip-conjugate.

We build T as an odomutant.

Reminder: $\psi \circ T = S \circ \psi$ almost everywhere

$$\begin{array}{cccl} \psi\colon & X & \longrightarrow & X \\ & (x_n)_{n\geq 0} & \longmapsto & \left(\sigma_{x_1}^{(0)}(x_0),\, \sigma_{x_2}^{(1)}(x_1),\, \sigma_{x_3}^{(2)}(x_2), \ldots\right) = \lim_{n\to+\infty} \psi_n(x) \end{array}$$

Sketch of proof of:

Theorem (C. 2025+)

Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear map and let S be an odometer. Then there exists $T \in \mathrm{Aut}(X,\mu)$ such that T and S are φ-OE but not flip-conjugate.

We build T as an odomutant.

Reminder: $\psi \circ T = S \circ \psi$ almost everywhere

$$\begin{array}{rccl} \psi\colon & X & \longrightarrow & X \\ & (x_n)_{n\geq 0} & \longmapsto & \left(\sigma^{(0)}_{x_1}(x_0),\, \sigma^{(1)}_{x_2}(x_1),\, \sigma^{(2)}_{x_3}(x_2), \dots\right) = \lim_{n\to+\infty} \psi_n(x) \end{array}$$

If ψ is invertible, then T and S are isomorphic (by definition)

Question (Coalescence property)

If T and S are isomorphic, is ψ invertible?

Yes! because S is an odometer (**No!** in full generality)
It holds more generally when S is ergodic and has discrete spectrum.

Definition

Let $S \in \mathrm{Aut}(X, \mu)$. An *eigenfunction* of S is $f \in \mathrm{L}^2(X, \mu) \setminus \{0\}\}$ such that $f \circ S = \lambda f$ for some $\lambda \in \mathbb{C}$ (*eigenvalue*).
$E_\lambda(S) \coloneqq \{\text{eigenfunctions associated to } \lambda\} \cup \{0\}$ (eigenspace).

If S is ergodic, then for every eigenvalue λ, $E_\lambda(S)$ is a line.

Definition

$S \in \mathrm{Aut}(X, \mu)$ has *discrete spectrum* if the span of all of its eigenfunctions is dense in $\mathrm{L}^2(X, \mu)$. (example: odometers)

Property

If S is ergodic and has discrete spectrum, then S is coalescent: for every $T \in \mathrm{Aut}(X, \mu)$ isomorphic to S, every factor map ψ from T to S is invertible.

Proof: $f \in E_\lambda(S) \Rightarrow f \circ \psi \in E_\lambda(T)$
By ergodicity, $E_\lambda(T) = E_\lambda(S) \circ \psi$
T and S have discrete spectrum: $\mathrm{L}^2(X, \mu) = \mathrm{L}^2(X, \mu) \circ \psi$.
So ψ is invertible.

Sketch of proof of the theorem:

We build T as an odomutant such that

- ψ is not invertible (T and S are not flip-conjugate);
- the distortions of the orbits are moderate (φ-integrable cocycles)